Exercícios sobre o modelo binomial e o modelo Black-Scholes do livro do Hull

10.15. A stock price is currently \$40. Over each of the next two three-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% per annum with continuous compounding.

a. What is the value of a six-month European put option with a strike price of \$42?

b. What is the value of a six-month American put option with a strike price of \$42?

10.18. A stock price is currently \$30. Each month for the next two months it is expected to increase by 8% or reduce by 10%. The risk-free interest rate is 5%. Use a two-step tree to calculate the value of a derivative that pays off $\max[(30 - S_T)^2, 0]$, where S_T is the stock price in two months? If the derivative is American-style, should it be exercised early?

10.19. Consider a European call option on a non-dividend-paying stock where the stock price is \$40, the strike price is \$40, the risk-free rate is 4% per annum, the volatility is 30% per annum, and the time to maturity is six months.

a. Calculate u, d, and p for a two-step tree.

b. Value the option using a two-step tree.

c. Verify that DerivaGem gives the same answer.

d. Use DerivaGem to value the option with 5, 50, 100, and 500 time steps.

11.13. A company's cash position (in millions of dollars) follows a generalized Wiener process with a drift rate of 0.1 per month and a variance rate of 0.16 per month. The initial cash position is 2.0.

a. What are the probability distributions of the cash position after one month, six months, and one year?

b. What are the probabilities of a negative cash position at the end of six months and one year?

c. At what time in the future is the probability of a negative cash position greatest?

11.14. Suppose that x is the yield on a perpetual government bond that pays interest at the rate of \$1 per annum. Assume that x is expressed with continuous compounding, that interest is paid continuously on the bond, and that x follows the process

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price? What is the expected instantaneous return (including interest and capital gains) to the holder of the bond?

10.15 Figure 1 displays the binomial tree that describes the value of the put options as a function of the stock price. At each node, the upper number refers to the stock price, whereas figures within parentheses and brackets correspond to the prices of the European and American put options, respectively. The risk-neutral probability p of an upward movement is

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{\frac{1}{4}0.12} - 0.9}{1.1 - 0.9} = 0.6523$$

given that the risk free rate of return is 12%. As the value of the European option at node A coincides with the present value of the expected payoff, it follows that

$$\begin{aligned} f &= \left[f_{uu}\,p^2 + 2\,f_{ud}\,p\,(1-p) + f_{dd}\,(1-p)^2\right]\,e^{-\frac{1}{2}r} \\ &= \left[2\,(42 - 39.6) \times 0.6523 \times 0.3477 + (42 - 32.4) \times 0.3477^2\right]\,e^{-\frac{1}{2}0.12} = 2.118 \end{aligned}$$

Needless to say, the same result holds if one works back the value of the option at node A through the tree. As for the American put option, it has a greater value than the European option because it is optimal to exercise early at node C. To appreciate that, it suffices to observe that the value of the European put option at node C is

$$\begin{aligned} f_d &= \Big[f_{ud}\,p + f_{dd}\,(1-p)\Big]\,e^{-\frac{1}{4}r} \\ &= \Big(2.4 \times 0.6523 + 9.6 \times 0.3477\Big)\,e^{-\frac{1}{4}0.12} = 4.759, \end{aligned}$$

whereas the payoff from early exercise is $K - S = 42 - 36 = 6$.

10.18 The risk-neutral probability of a upward movement is

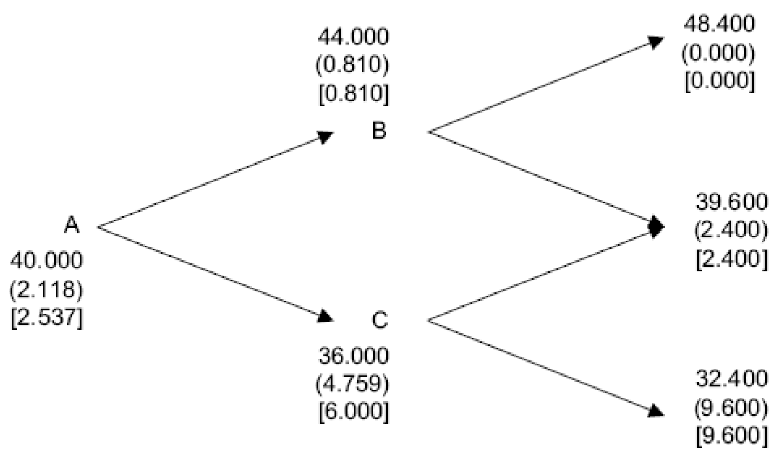
$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{\frac{1}{12}0.05} - 0.9}{1.08 - 0.9} = 0.5788,$$

hence it follows from the present value of the expected payoff at node A in Figure 2 that the value of the power option is

$$\begin{aligned} f &= \left[f_{uu}\,p^2 + 2\,f_{ud}\,p\,(1-p) + f_{dd}\,(1-p)^2\right]\,e^{-\frac{1}{6}r} \\ &= \left[2\,(30 - 29.16)^2 \times 0.5788 \times 0.4212 + (30 - 24.3)^2 \times 0.4212^2\right]\,e^{-\frac{1}{6}0.05} = 6.0587 \end{aligned}$$

As before, the same result holds if one backs out the value of the power option by working back through the binomial tree below. Finally, the result also holds even for the American options in view that, at node C, the payoff from early exercise is $(K - S)^2 = (30 - 27)^2 = 9$, which is less

Figure 1



than the value of the analogous European power option, namely

$$\begin{aligned} f_d &= \left[f_{ud}\, p + f_{dd}\,(1-p) \right] e^{-\frac{1}{12} r} \\ &= \left[(30-29.16)^2 \times 0.5788 + (30-24.3)^2 \times 0.4212 \right] e^{-\frac{1}{12} 0.05} = 14.0346 \end{aligned}$$

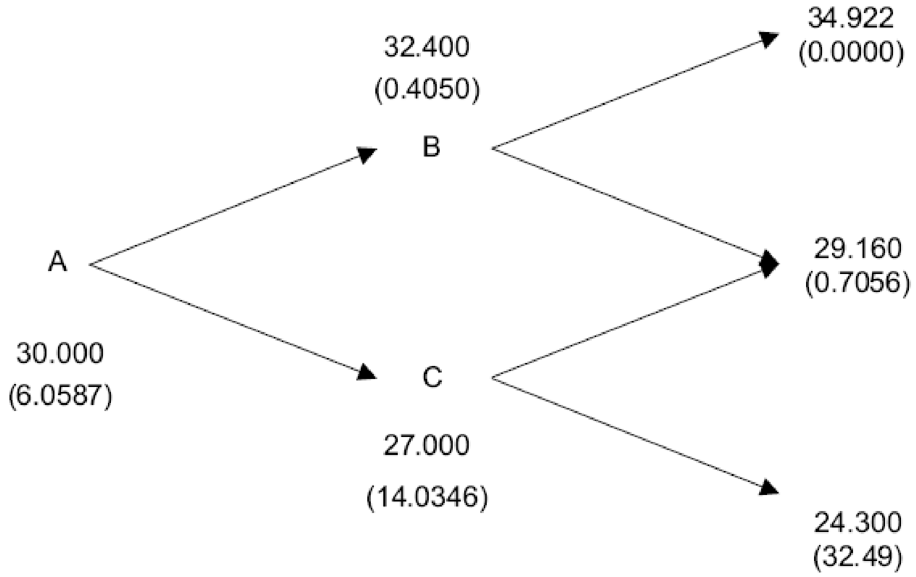
10.19 The binomial tree has two steps and time to maturity of the option is 6 months, hence each step will last for three months, i.e., $\delta t = 1/4$. As the volatility is 30% per annum, it follows that $u = e^{\sigma\sqrt{\delta t}} = e^{\frac{1}{2} 0.3} = 1.1618$ and $d = 1/u = 0.8607$, resulting in a risk-neutral probability of a upward movement of

$$p = \frac{e^{r\,T} - d}{u - d} = \frac{e^{\frac{1}{3} 0.04} - 0.8607}{1.1618 - 0.8607} = 0.4959$$

The value of the option then is 3.3739 as calculated by DerivaGem (select Equity as the underlying type and Binomial European as the option type). If one uses 5, 50, 100 and 500 time steps, DerivaGem will compute the value of the option as 3.9229, 3.7394, 3.7478, and 3.7545, respectively.

11.13 If the cash position follows a generalized Wiener process with a drift rate of 0.1 per month and a variance rate of 0.16 per month with an initial position of 2, then the probability distribution of the cash position after k months is normal with mean $2{+}0.1\,k$ and variance $0.16\,k$. The probability of a random sample from a normal distribution with mean 2.6 and variance 0.96 being negative is $\Phi\left(-\frac{2+6\times 0.1}{\sqrt{6\times 0.16}}\right) = 0.0040$, where Φ is the cumulative distribution function of the standard normal. As for the one-year horizon, the probability of negative cash position is $\Phi\left(-\frac{2+12\times 0.1}{\sqrt{12\times 0.16}}\right) = 0.0107$. Finally, to find at what time the probability of a negative cash flow is greatest, it suffices to minimize

Figure 2



$\frac{2+0.1\,k}{\sqrt{0.16\,k}}$ with respect to k. The first derivative is zero when $k = 20$ months (with positive second derivative).

11.14 By Ito's lemma, the bond price will follow

$$\mathrm{d}B = \left[\frac{\partial B}{\partial x}\,a\,(x_0 - x) + \frac{\partial B}{\partial t} + \frac{\partial^2 B}{\partial x^2}\,s^2\,x^2\right]\mathrm{d}t + \frac{\partial B}{\partial x}\,s\,x\,\mathrm{d}z.$$

The price of a perpetual bond that continuously pays interest at the rate of \$1 per annum is $B = \int_0^\infty e^{-x\,t}\,\mathrm{d}t = 1/x$. It then follows that $\frac{\partial B}{\partial x} = -x^{-2}$, $\frac{\partial B}{\partial t} = 0$, and $\frac{\partial^2 B}{\partial x^2} = 2\,x^{-3}$. The stochastic process of the bond price thus is

$$\mathrm{d}B = \left[-a\,\frac{x_0 - x}{x^2} + \frac{s^2}{x}\right]\mathrm{d}t - \frac{s}{x}\,\mathrm{d}z,$$

which means that the overall expected instantaneous return (i.e., interest plus the expected capital gains) is $1 - a\,\frac{x_0 - x}{x^2} + \frac{s^2}{x}$.